

Colliding Singularities in F-theory and Phase Transitions

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ABSTRACT

We study F-theory on elliptic threefold Calabi-Yau near colliding singularities. We demonstrate that resolutions of those singularities generically correspond to transitions to phases characterized by new tensor multiplets and enhanced gauge symmetry. These are governed by the dynamics of tensionless strings. We also find new transition points which are associated with several small instantons simultaneously shrinking to zero size.

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1. Introduction

Great progress has been made recently in our understanding of enhanced gauge symmetries and matter contents in compactifications of F-theory [1] [2] [3] [4]. In this paper we further develop the *geometry/physics* dictionary for some F-theory compactifications on elliptic Calabi-Yau threefolds. Such an analysis can provide us with additional evidence for string-string duality as well as give us a better understanding of quantum field theory applications of the latter [5]. Only elliptic Calabi-Yau manifolds appear in F-theory compactifications. The elliptic fibration is defined over some two-dimensional base B and can be written in Weierstrass form

$$y^2 = x^3 + xf(z, w) + g(z, w) , \quad (1.1)$$

where (z, w) parametrize two-dimensional base. The appearance of singularities in elliptic fibration is responsible for the enhanced gauge symmetry. The singularities in the elliptic fiber were classified by Kodaira and are summarized in the table below.

Table 1: Kodaira Classification of Singularities

$\text{ord}(f)$	$\text{ord}(g)$	$\text{ord}(\Delta)$	fiber type	singularity type
≥ 0	≥ 0	0	smooth	none
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Only the local structure of singularity around the degeneration divisor (position of the 7-brane) is relevant. The physical reasoning for appearance enhanced gauge symmetry is clear. Open strings with various (p, q) charges connecting 7-branes [6] [1] [7] [8] [9] become massless vector particles and promote abelian symmetry to non-abelian one. Various singularities of F-theory compactification were analyzed in ref. [3] and corresponding gauge groups have been found, including the non-simply laced series B and C as well as F_4 and G_2 . It appears that generic singularity does not usually correspond to the maximal gauge group associated with it [3][10]. The reason for this is that there are certain monodromies along the curve of singularities given by either internal (split singularity) or outer (nonsplit singularity) automorphisms of the root lattice. It has also been found that

the gauge groups associated with two intersecting 7-branes cannot have simultaneously a perturbative explanation from the heterotic point of view. In the usual setup one of the gauge groups is perturbative while the other one is realized via small instantons shrinking to zero size on the heterotic side. We are going to call the two intersecting D-branes with the gauge groups as “collision of the gauge groups”, borrowing this term from mathematics (“collision of singularities”). This theory is nonanomalous only if the intersecting 7-branes correspond to either two SU ’s or SU and SO gauge groups. One can analyse these types of collisions of singularities following the Katz-Vafa suggestion [4]. The basic idea is very simple. Fiberizing 8-dimensional theory with gauge symmetry \mathcal{G} , one obtains 6-dimensional compactification. The fibration parameter t can be interpreted as the vev of the adjoint scalar. For $t \neq 0$ the theory possesses $G \times G' \times U(1) \subset \mathcal{G}$ symmetry. The spectrum of 6-dimensional theory follows unambiguously from the Higgs mechanism. This method works only if one can break $\mathcal{G} \longrightarrow G \times G' \times U(1)$ by giving vev to adjoint matter. Indeed, groups $SU(n) \times SU(m) \times U(1)$ or $SU(n) \times SO(2m) \times U(1)$ can be obtained by breaking down $SU(n+m)$ or $SO(2m+2n)$, respectively. It is easy to explain this process as splitting a Dynkin diagram into two parts by removing one node (which corresponds to $U(1)$ factor). Therefore, the collisions of gauge groups whose Dynkin diagrams cannot be embedded into bigger one cannot be analyzed in this way.

The simplest example where one cannot find the spectrum of the theory is the collision of $SO(n)$ groups. It is impossible to satisfy the anomaly cancellation conditions for F-theory. It seems that the local field theory description does not exist.

To be specific, suppose that a discriminant locus contains two intersecting D-branes, say D and D' , each corresponding to the gauge groups $SO(2n+8)$. From the mathematical point of view, the Calabi-Yau threefold with two colliding I_n^* singularities (I_n^* singularity corresponds to $SO(2n+8)$) appears to be very singular and requires a resolution. Resolving the singularity it is not enough to blow up a fiber and one needs also to blow up a base at the point of intersection. That means replacing an intersection point by the whole P^1 and, as the result, the divisors D and D' do not intersect each other on the blown up surface. It happens very often that the new sphere becomes also a component of the discriminant locus. In this situation one may get a nonperturbative gauge symmetry enhancement. The new gauge group G is associated with the blown up divisor and is determined by the structure of elliptic fibration. It appears that after the resolution one can also satisfy the anomaly factorization conditions.

From the physical point of view this collision can be understood as follows. One can start with perfectly well defined theory with two intersecting D-branes carrying, say for definiteness, gauge groups $SO(*) \times SU(*)$. By adjusting the vev ’s of the hypermultiplets one can enhance the second group to $SO(*)$, keeping the first one intact. At this very moment we reach the transition point and the local field theory description becomes inconsistent. A new branch with an additional tensor multiplet, known as “Coulomb” branch [11], is attached at the transition point. It is characterized by the vacuum expectation value of the scalar field in the tensor multiplet. This branch becomes the standard

Coulomb branch after compactification down to 4-dimensions. The transition point is governed by the dynamics of the tensionless strings [11][12] [13] [14] [15]. The appearance of tensionless strings is clear if we approach the transition point from the Coulomb branch. The tensionless strings correspond to 3-branes wrapped around the vanishing (blown up or down) 2-cycle. The tension of the strings is given by the vev of tensor multiplet, or expressing in mathematical terms, by the area of the blown up sphere. It is also worth mentioning that after the phase transition, the theory ceases to have heterotic dual.

In the Coulomb branch, we get a perfectly nonanomalous theory. In some cases the gauge symmetry gets enhanced. The gauge coupling constant of this new nonperturbative symmetry is governed by the vev of the tensor multiplet

$$\frac{1}{g^2} \sim vev \sim \text{area}(P^1) . \quad (1.2)$$

Approaching the transition point from the Coulomb branch one recovers a strong coupling transition. This also requires a fine tuning of both hypermultiplets and tensor multiplets. It is worth mentioning that passing through the transition point to the Coulomb branch makes some of the original hypermultiplets heavy. In what follows we will be able to analyze various collisions and to predict corresponding gauge groups and their matter contents.

Some of the phase transitions discussed in this paper are known to be associated to a single small E_8 instanton. We also find new phase transition points related to two, three and five small E_8 instantons collapsing simultaneously.

Organization of this paper is as follows: we first review the anomaly factorization condition for F-theory compactifications. In section 3 we present mathematical discussion of the blowup procedure. In section 4 we discuss the physical interpretations of the collisions using the blowup technique. Finally, we conclude with an analysis of the role of small E_8 instantons and with the discussion of new phenomena whose interpretation in terms of tensionless strings is unclear.

2. Anomaly cancellation and colliding singularities

Here we will review the conditions necessary for anomaly cancellation in 6-dimensions. Anomaly cancelation via Green-Schwarz mechanism [16] [17] requires that a certain 8-form should be factorized. This implies that at least the coefficient in front of $\text{tr} R^4$ should vanish, which is equivalent to some relation between the number of n_V (vector), n_H (hyper) and n_T (tensor) multiplets:

$$n_H - n_V + 29n_T = 273 . \quad (2.1)$$

The resolution of the singularities changes the number of tensor multiplets. Only the local structure of the singularity is relevant and therefore this implies $\delta n_H - \delta n_V + 29 = 0$ for each blowup.

There are other conditions that also should be satisfied in order for factorization to take place. The conditions for anomaly cancellation in F-theory has been found in ref. [18]. Let us denote the irreducible components of the discriminant locus as $D_a \subset B$. Each component of the discriminant locus D_a gives rise to the gauge group G_a , depending on the singularity in the elliptic fibration. The gauge fields propagate inside 7-branes wrapped around divisors D_a .

The anomaly factorization conditions are

$$\begin{aligned}
\sum_{(R_a, R_b)} n_{(R_a, R_b)} \text{Ind}(R_a) \text{Ind}(R_b) &= (D_a \cdot D_b), \\
\text{Ind}(Ad_a) - \sum_R \text{Ind}(R_a) n_{R_a} &= 6(K \cdot D_a), \\
y_{Ad_a} - \sum_R y_{R_a} n_{R_a} &= -3(D_a \cdot D_a), \\
x_{Ad_a} - \sum_R x_{R_a} n_{R_a} &= 0.
\end{aligned} \tag{2.2}$$

Here, $\text{Ind}(R)$ stands for the Dynkin index of representation R , the numbers n_{R_a} denote a multiplicity of the representation R_a of the gauge group G_a , and $n_{(R_a, R_b)}$ denotes the number of mixed representations. Parameters x_R and y_R are defined by the following decomposition

$$\text{tr}_R F^4 = x_R \text{tr} F^4 + y_R (\text{tr} F^2)^2, \tag{2.3}$$

assuming R has two independent order four invariants; tr_R denotes the trace in the representation R , and tr stands for the trace in some standard representation (usually fundamental representation).

Let us explain how the local field theory “feels” the collision of singularities and why one should resolve it. Suppose that discriminant locus contains two intersecting D-branes, say D and D' , each corresponding to the gauge groups G and G' . The local analysis of D-branes [19] [18] implies that each intersection point should give rise to hypermultiplets in mixed representation (R, R') . Moreover, one can discuss the anomaly cancellation locally at each intersection point (the first equation in (2.2)). The condition for having a consistent theory reads that either $n_{(R, R')} = \text{Ind}(R) = \text{Ind}(R') = 1$ or $n_{(R, R')} = 1/2$, and indices $\text{Ind}(R) = 1$ and $\text{Ind}(R') = 2$ (the latter solution is possible only for pseudoreal representation (R, R')). Now we can see that the case $SO(n) \times SO(m)$ is completely ruled out because all indices $\text{Ind} \geq 2$ for $n, m \geq 7$.

3. Blowups

One of the main tools in resolution of singularities is the blowing up procedure [20]. We will discuss here a blowing up of an algebraic surface at a point P . We parametrize an open neighborhood $U_P \subset B$ using affine coordinates (z, w) . One can think about U_P being

embedded in A^2 (affine two-dimensional space). Consider the product $A^2 \times P^1$, which is a quasi projective variety, with (y_1, y_2) being homogeneous coordinates in P^1 . Define the *blowing up* of B at point $P = (0, 0)$ as the closed subset in $\hat{U}_P \subset A^2 \times P^1$, defined by the equations

$$zy_2 = wy_1 . \quad (3.1)$$

We will denote the blown up surface by \hat{B} .

Let us denote the map $\pi : U_P \times P^1 \rightarrow U_P$, defined by the above equation. Then $\pi^{-1}(Q)$ for $Q \neq P$ consists of one point, while $\pi^{-1}(P) = P^1$. In other words the blowup procedure “replaces” the point $P = (0, 0)$ by P^1 . This P^1 is called an exceptional divisor and we reserve the notation E for it. To parameterize the blown up base around the point P one can use either $(z, \xi = y_2/y_1)$ or $(w, \eta = y_1/y_2)$. The map $\pi_* : H^2(\hat{B}, \mathbf{Z}) \rightarrow H^2(B, \mathbf{Z})$ respects the intersection form and has a kernel generated by E . Denote by D^* the full preimage of a divisor D in \hat{B} . Suppose that the divisor D passes through the point P we are blowing up. In this case the divisor D^* happens to be reducible. Namely, $D^* = \hat{D} + E$, where both \hat{D} and E are irreducible. Suppose two divisors D and D' intersect each other at the point P . In the blown up surface \hat{B} the irreducible divisors \hat{D} and \hat{D}' do not intersect each other

$$\begin{aligned} (\hat{D} \cdot \hat{D}') &= ((D^* - E) \cdot (D'^* - E)) = 0 \\ (\hat{D} \cdot E) &= 1 , \quad (\hat{D}' \cdot E) = 1 . \end{aligned} \quad (3.2)$$

Intersection pairing of the divisors that do not pass through the intersection point remains unchanged. The canonical class of the blown up surface is equal to $\hat{K} = K^* + E$.

The elliptic fibration can be pulled up on the blown up surface \hat{B} . The new fibration is defined as

$$\begin{aligned} f(z, w) &\longrightarrow f(z, \xi) = f(z, z\xi) \\ g(z, w) &\longrightarrow g(z, \xi) = g(z, z\xi) . \end{aligned} \quad (3.3)$$

For our purpose having elliptic fibration is not enough. One has to check whether the elliptic fibration on the blown up surface determines the Calabi-Yau manifold. The condition for having Calabi-Yau is that canonical class can be written in terms of the irreducible components of the discriminant locus

$$K(B) = - \sum_i a_i [D_i] - a[D] - a'[D'] , \quad (3.4)$$

where we explicitly singled out two intersecting components of the discriminant locus. Upon the blowup the divisor E becomes a new component of the discriminant. The relation for the blown up surface reads

$$K^*(B) + [E] = - \sum_i a_i [D_i^*] - a[\hat{D}] - a'[\hat{D}'] - b[E] , \quad (3.5)$$

where b is determined by the type of singularity of the fiber over the blown up sphere. In order for these two relations to be satisfied it is necessary that

$$a + a' - b = 1 . \quad (3.6)$$

This relation severely constrains possible collisions, which admit resolutions by blowing up the base and preserve the Calabi-Yau condition (3.6). The coefficients a_i are summarized in the table below.

Table 2. Coefficients a_i .

none	I_n	II	III	IV	I_n^*	IV^*	III^*	II^*
0	$\frac{n}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2} + \frac{n}{12}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$

It is worth mentioning that there are several collisions, known as dual, for which $a + a' = 1$, $b = 0$. These collisions do not lead to any enhanced gauge symmetry.

To make blowup one has to adjust an appropriate number of parameters. Blowup does not introduce any new complex parameters. To be able to pull up the elliptic fibration on the blown up base the Calabi-Yau manifold should be described by the following Weierstrass form [2]

$$y^2 = x^3 + x \sum_{l+k \geq 4} z^l w^k f_{l,k} + \sum_{l+k \geq 6} z^l w^k g_{l,k}. \quad (3.7)$$

We assumed that the base of the elliptic fibration is Hirzebruch surface. Condition (3.6) appears to be equivalent to the restrictions on the coefficients in the expansion (3.7). The blowup point $(z, w) = (0, 0)$ lies on the discriminant locus, which in general may correspond to a very mild singularity, say I_1 . This singularity does not lead to any gauge symmetry enhancement. We mostly, are going to be interested in different situations, when the blowup point coincides with the intersection of two divisors, each corresponding to nonabelian gauge symmetry. We will return back to this discussion in the last section.

4. The cases

As we discussed in the introduction, collisions of singularities lead to singular Calabi-Yau manifolds. To resolve these singularities, one needs to make an appropriate number of blowups. In this section we discuss only those collisions which can be blown up without violation of Calabi-Yau condition (3.6). The allowed collisions of singularities can be classified by the value of the modular invariant function $J(\tau)$ at the collision point [21]. Only the fibers with the same value of $J(\tau)$ at the intersection point can collide. Some “collisions” of singular elliptic curves produce singular Calabi-Yau threefolds that can be resolved by blowing up a fiber (small resolutions). For example, the collisions $I_n \times I_m$

$(SU(*) \times SU(*))$ and $I_n \times I_m^*$ ($SU(*) \times SO(*)$) are exactly of this type (considered in [3][4]).

Other collisions do not have one-dimensional resolutions. These situations can be resolved by suitable blowing up the base at the collision point and pulling up the fibration to the blown up surface. In some cases one has to repeat the blowup operation several times. In this process one creates more collisions. The whole procedure stops when all collisions allow small resolutions and can be resolved by blowing up a fiber.

At this point it is instructive to compare the physical interpretation of “blowing up” a fiber with that of “blowing up” a base. The blowup of a fiber becomes visible upon compactification down to 5 dimensions. The area of the blown up sphere coincides with the *vev* of the scalar field, which parametrizes the Coulomb branch in 5 dimensions. The blowup of the base is already visible in 6 dimensions and corresponds to the appearance of a new tensor multiplet.

In the examples considered below, we usually end up with enhanced gauge symmetry, say $G_1 \times \mathcal{H} \times G_2$, upon the resolution. The gauge groups G_1 and G_2 correspond to colliding singularities and in some sense serve as our “initial data”. The intermediate factor \mathcal{H} describes the gauge symmetry enhancement. The matter content is given by hypermultiplets in various representations. The part of the hypermultiplet spectrum, charged only with respect to either G_1 or G_2 , depends on particular details of the theory, such as the choice of the base of the elliptic fibration and the choice of the divisors. The matter in \mathcal{H} representation as well as in the mixed representations of $G_{1,2} \times \mathcal{H}$ are *universal* and depend only on the local structure of the colliding singularities.

In order to be specific and make the examples more transparent, we need to specify some details of the theory. For simplicity we assume that intersecting divisors are homologically spheres and their intersections are: $D \cdot D = n_1$, $D' \cdot D' = n_2$ and $D \cdot D' = 1$. After the blowup the irreducible components \hat{D} and \hat{D}' have the following intersection pairing

$$\begin{aligned} \hat{D} \cdot \hat{D} &= n_1 - 1, & \hat{D}' \cdot \hat{D}' &= n_2 - 1, & E \cdot E &= -1, \\ \hat{K} \cdot \hat{D} &= -n_1 - 1, & \hat{K} \cdot \hat{D}' &= -n_2 - 1, & \hat{K} \cdot E &= -1 \quad . \end{aligned} \tag{4.1}$$

For example, in the case of Hirzebruch surface \mathbf{F}_n , if the singularities are along $D = D_u$ (the base) and $D' = D_s$ (the P^1 fiber), then n_1 , n_2 are equal to n and 0 or vice versa.

4.1. Dual singularities

The case $I_0^* \times I_0^*$ is the first example of the collision of dual fibers. It is also special because it can be realized for any value of J . Without the second singularity each of these degenerations gives rise to the consistent theory with maximal allowed gauge group $SO(8)$ (which corresponds to split singularity), each having $(n_i + 4)(\mathbf{8}_v + \mathbf{8}_c + \mathbf{8}_s)$ matter hypermultiplets. As it was noticed in [3], when the singularities collide, one can not get a consistent local field theory description.

To resolve such a singularity one needs to perform a blowup of an intersection point P . For dual singularities the exceptional divisor E is *not* a component of the discriminant locus. The fiber over E is smooth and there is no gauge symmetry enhancement. So the resolution of the paradox is, that on the blown up base, the divisors \hat{D} and \hat{D}' do not meet each other and there are no mixed representations. Going through the analysis of the anomaly equations (2.2) or codimensional counting, we conclude that in the new branch (with non zero *vev* of the tensor multiplet) there is still a gauge group $SO(8) \times SO(8)$ with $(n_i + 3)(\mathbf{8}_v + \mathbf{8}_c + \mathbf{8}_s)$ matter hypermultiplets for each factor ³. The change in the matter spectrum $(n_i + 4) \rightarrow (n_i + 3)$ can be interpreted as the result of some coupling between the matter hypermultiplets and the tensor multiplet (some hypermultiplets become heavy).

In fact there is nothing special about $SO(8)$ gauge group. It corresponds to I_0^{*s} singularities. We can also discuss $SO(7)$ (I_0^{*ss}) or G_2 (I_0^{*ns}) singularity. Indeed, by giving expectation value to two spinors, we can higgs $SO(8)$ down to G_2 . The case of G_2 is interesting: it corresponds to nonsplit singularity and one can immediately count the dimension of the moduli space of hypermultiplets. The collision $I_0^{*ns} \times I_0^{*ns}$ corresponds to $G_2 \times G_2$ gauge group with the matter contents $(3n_i + 7) \mathbf{7}$ without any *mixed* matter.

For simplicity we present the parameter counting for $P^1 \times P^1$ being the base ($n_1 = n_2 = 0$), but this computation can be generalized for any Hirzebruch surface. It follows from the structure of polynomials f, g , that the number of independent parameters is $7^2 + 10^2 - 4 = 145$ ⁴. The total number of $n_H - n_V = 273 - 29n_T$. After the blowup $n_T = 2$ and, as a result, we get a perfect match

$$145 + 7 \cdot 7 + 7 \cdot 7 - 14 - 14 = 215. \quad (4.2)$$

The counting for other cases $I_0^* \times I_0^*$ is obvious because they are related to $G_2 \times G_2$ by Higgs mechanism.

There are several cases like the above one, namely, $II \times II^*$, $III \times III^*$ and $IV \times IV^*$. In all these cases the exceptional divisor E does not belong to the discriminant locus and therefore there is no gauge symmetry enhancement. The discussion is very similar to the $I_0^* \times I_0^*$ collision. For colliding singularities these gauge theories are anomalous because they should necessarily have matter in the mixed representations. For the blown up base the divisors \hat{D} and \hat{D}' do not intersect and therefore in this phase the theory does not have any mixed representations and one can satisfy the anomaly factorization conditions. We summarize all these results in the table below.

³ This result is consistent with the examples discussed in refs. [2] [22] [23] in the case of some orientifold compactifications.

⁴ $4 = 6 - 2$ comes from $SL(2, \mathbf{Z}) \times SL(2, \mathbf{Z})$ residual symmetry. Each P^1 has a marked point (location of singularity), which explains -2 .

Table 3. Gauge groups and matter for collisions of dual singularities.

Singularity	Gauge group	Matter
$I_0^{*ns} \times I_0^{*ns}$	$G_2 \times G_2$	$(3n_1 + 7)(\mathbf{7}, \mathbf{1}) + (3n_2 + 7)(\mathbf{1}, \mathbf{7})$
$II \times II^*$	E_7	$\frac{1}{2}(n_1 + 7)\mathbf{56}$
$III \times III^*$	$SU(2) \times E_7$	$\frac{1}{2}(n_1 + 7)(\mathbf{1}, \mathbf{56}) + (6n_2 + 10)(\mathbf{2}, \mathbf{1})$
$IV^s \times IV^{*ns}$	$SU(3) \times F_4$	$(n_1 + 4)(\mathbf{1}, \mathbf{26}) + (6n_2 + 12)(\mathbf{3}, \mathbf{1})$

4.2. $J = \infty$ collisions

We first consider the collisions of $I_n^* \times I_m^*$ ($SO(*) \times SO(*)$) singularities with $n + m > 0$. In all these cases one has to blow up the base only once and the exceptional divisor E becomes the component of the *discriminant locus*. Therefore, we should expect the enhancement of gauge symmetry. The fiber over E generically corresponds to I_{m+n} (A_{n+m-1}) singularity and jumps to an I_m^* (I_n^*) singularity at the points of intersection with the D and D' divisors. We claim that for $n + m$ even the singularity along E is nonsplit and it corresponds to $Sp((n + m)/2)$ gauge symmetry⁵. There are three different cases to consider depending on whether the colliding singularities are split or nonsplit. All these cases differ from each other in the matter content. We summarized the results for case $n + m$ being even in the table below

Table 4. Gauge groups and universal matter for $J = \infty$ collisions for even $m + n$.

Singularity	Group, Universal Matter
$I_n^{s*} \times I_m^{s*}$	$SO(2n + 8) \times Sp((n + m)/2) \times SO(2m + 8)$ $\frac{1}{2}(\mathbf{2n} + \mathbf{8}, \mathbf{n} + \mathbf{m}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{n} + \mathbf{m}, \mathbf{2m} + \mathbf{8})$
$I_n^{ns*} \times I_m^{s*}$	$SO(2n + 7) \times Sp((n + m)/2) \times SO(2m + 8)$ $\frac{1}{2}(\mathbf{2n} + \mathbf{7}, \mathbf{n} + \mathbf{m}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{n} + \mathbf{m}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{n} + \mathbf{m}, \mathbf{2m} + \mathbf{8})$
$I_n^{ns*} \times I_m^{ns*}$	$SO(2n + 7) \times Sp((n + m)/2) \times SO(2m + 7)$ $\frac{1}{2}(\mathbf{2n} + \mathbf{7}, \mathbf{n} + \mathbf{m}, \mathbf{1}) + (\mathbf{1}, \mathbf{n} + \mathbf{m}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{n} + \mathbf{m}, \mathbf{2m} + \mathbf{7})$

It is remarkable, that in all these cases one can not break $Sp((n + m)/2)$ by giving expectation value to $(\mathbf{1}, \mathbf{4}, \mathbf{1})$. We present here only the *universal* part of the matter spectrum, which depends on the local structure of the singularity.

⁵ In principle we have to choose between $SU(n + m)$ and $Sp((n + m)/2)$, which is the choice between split and nonsplit cases. However, the split case does not satisfy the anomaly factorization condition.

From the mathematical point of view the case of $(n + m)$ odd is unclear. The group $SU(n + m)$ does not satisfy the anomaly conditions (2.2), while the nonsplit I_{2k+1} singularities does not seem to correspond to any gauge groups (at least the gauge groups are unknown [3]). Nevertheless, one can make some predictions based on the Higgs mechanism. It follows from the Higgs mechanism that in case of $n + m$ being odd the gauge group also gets enhanced by a factor $Sp((n + m)/2)$. The conjectured answers for $n + m$ being odd are summarized in the table below

Table 5. Gauge groups and universal matter for $J = \infty$ collisions for odd $m + n$.

Singularity	Group, Universal Matter
$I_n^{s*} \times I_m^{s*}$	$SO(2n + 8) \times Sp((n + m + 1)/2) \times SO(2m + 8)$ $\frac{1}{2}(\mathbf{2n} + \mathbf{8}, \mathbf{n} + \mathbf{m} + \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{n} + \mathbf{m} + \mathbf{1}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{n} + \mathbf{m} + \mathbf{1}, \mathbf{2m} + \mathbf{8})$
$I_n^{ns*} \times I_m^{s*}$	$SO(2n + 7) \times Sp((n + m + 1)/2) \times SO(2m + 8)$ $\frac{1}{2}(\mathbf{2n} + \mathbf{7}, \mathbf{n} + \mathbf{m} + \mathbf{1}, \mathbf{1}) + \frac{3}{2}(\mathbf{1}, \mathbf{n} + \mathbf{m} + \mathbf{1}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{n} + \mathbf{m} + \mathbf{1}, \mathbf{2m} + \mathbf{8})$
$I_n^{ns*} \times I_m^{ns*}$	$SO(2n + 7) \times Sp((n + m - 1)/2) \times SO(2m + 7)$ $\frac{1}{2}(\mathbf{2n} + \mathbf{7}, \mathbf{n} + \mathbf{m} - \mathbf{1}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{n} + \mathbf{m} - \mathbf{1}, \mathbf{2m} + \mathbf{7})$

Singularities I_n^* with n big enough can not be realized in compact Calabi-Yau space, because they destroy the triviality of canonical bundle. Nevertheless, it makes perfect sense to discuss the local structure of the resolution.

The cases with $n, m \leq 12$ can be realized as collisions in compact Calabi-Yau spaces. We start by considering the case $SO(12) \times SO(12)$, which corresponds to the collision of $I_2^* \times I_2^*$ singularities. The singularity along the exceptional divisor is A_3 (the type I_4 fiber). One can go through tedious calculations, imposing the conditions found in [3] and recover that $f(z, \xi)$ and $g(z, \xi)$ indeed corresponds to I_4 singularity (we do not present these calculations here). The real issue is whether we get $SU(4)$ or $SO(5)$. To answer this question, consider the leading behavior of $f(z, \xi)$ and $g(z, \xi)$

$$f(z, \xi) = -3(\xi h_0)^2 + O(z) , \quad g(z, \xi) = 2(\xi h_0)^3 + O(z) \quad (4.3)$$

Here, we definitely get a nonsplit singularity because the expansion for $f(z, \xi)$ starts as $h^2(\xi)$ (in the split case the expansion should start as $\tilde{h}^4(\xi)$).

This is consistent with the fact that we can find the solution of anomaly equations only for $SO(5)$ gauge group. The $SU(4)$ group requires matter in the representation $\frac{1}{2}(\mathbf{12}, \mathbf{4}, \mathbf{1})$, which does not make much sense. However this matter makes a perfect sense for $SO(5)$. Finally, the matter spectrum is given by $(n_i + 5) \mathbf{12} + \frac{1}{2}(n_i + 3) \mathbf{32}$ and the *universal* part

$$\frac{1}{2}(\mathbf{12}, \mathbf{4}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{4}, \mathbf{12}) . \quad (4.4)$$

It is remarkable that there is no matter charged only with respect to $SO(5)$. Now one can higgs $SO(12)$ down to $SO(n)$ for $8 \leq n < 12$ in order to “derive” the gauge groups and the matter contents for these cases. In doing this we find some surprises.

Consider giving expectation value to two vectors of $SO(12)$. In doing this we recover the collision of two groups $SO(10) \times SO(12)$. This is exactly the case for which we do not have any prediction, from ref. [3]. By blowing up this collision one should end up with the $SO(10) \times SO(5) \times SO(12)$ gauge group and matter in the $(n_1 + 3) (\mathbf{10}, \mathbf{1}, \mathbf{1}) + (n_1 + 3) (\mathbf{16}, \mathbf{1}, \mathbf{1}) + (n_2 + 5) (\mathbf{1}, \mathbf{1}, \mathbf{12}) + \frac{1}{2}(n_2 + 3) (\mathbf{1}, \mathbf{1}, \mathbf{32})$ representation, as well as the *universal* part

$$\frac{1}{2}(\mathbf{10}, \mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{4}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{4}, \mathbf{12}). \quad (4.5)$$

In spite of the fact that there is matter charged only with respect to nonperturbative $SO(5)$, one cannot higgs it. This case corresponds to I_3^{ns} (nonsplit). Therefore, we have a definite prediction that I_3^{ns} should correspond to $SO(5)$. Again, it is remarkable that the group $SU(3)$ is ruled out by the same arguments as $SU(4)$.

Going further down one can higgs the other $SO(12)$. In doing this we recover $SO(10) \times SO(5) \times SO(10)$ with the $(n_i + 3)(\mathbf{10} + \mathbf{16})$ and

$$\frac{1}{2}(\mathbf{10}, \mathbf{4}, \mathbf{1}) + 2(\mathbf{1}, \mathbf{4}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{4}, \mathbf{10}) \quad (4.6)$$

hypermultiplets. Now we have enough matter to break nonperturbative $SO(5)$ to $SU(2)$ without destroying both $SO(10)$. In this process two $(\mathbf{1}, \mathbf{4}, \mathbf{1})$ get eaten. As the result we get $(n_i + 4)\mathbf{10} + (n_i + 3)\mathbf{16}$ of each $SO(10)$ as well as *universal* mixed representations

$$\frac{1}{2}(\mathbf{1}, \mathbf{2}, \mathbf{10}) + \frac{1}{2}(\mathbf{10}, \mathbf{2}, \mathbf{1}) .$$

It is remarkable that there is no matter charged with respect to $SU(2)$.

These results can be compared with predictions coming from the collisions of singularities [21]. Namely, the singularity along the exceptional divisor should be of type I_2 , which corresponds to $SU(2)$ gauge symmetry enhancement. Imposing the condition for two $SO(10)$ we recover that

$$\begin{aligned} f(z, \xi) &= -3(\xi h_0)^2 + z(-6h_0 h_1 + f_{10})\xi^3 + O(z^2) \\ g(z, \xi) &= 2(\xi h_0)^3 - z(h_0 \xi)(-6h_0 h_1 + f_{10})\xi^3 + O(z^2) , \end{aligned} \quad (4.7)$$

which is exactly the condition for having $SU(2)$. On the other hand the Higgs mechanism predicts two solutions: $SO(5)$ gauge group with the matter and $SU(2)$ without matter. The resolution of this puzzle is that generically we get $SU(2)$, which gets enhanced to $SO(5)$ along some locus.

Higgsing down, one can obtain the results for $SO(9)$, $SO(8)$, $SO(7)$ or G_2 . It is worth mentioning that the classification of ref. [21] describes the *generic* singularities.

They correspond to the *minimal* possible gauge groups. For example, in the present discussion *generic* I_2 singularity appearing upon the resolution of $SO(10) \times SO(10)$ collision corresponds to $SU(2)$, for $SO(8) \times SO(8)$ collision there is generically no gauge group enhancement. Clearly, going to various subloci in the hypermultiplet moduli space, one can get different enhanced gauge symmetries. Namely, one can obtain $SO(8) \times SO(5) \times SO(8)$ with four matter hypermultiplets in $(\mathbf{1}, \mathbf{4}, \mathbf{1})$ representation.

4.3. $J = 1$ collisions

The only collision which can be resolved by blowing up a fiber is $III \times I_0^*$ ($A_1 \times D_4$).

There are two cases where a blowup of the base is required: these are $I_0^* \times III^*$, $III^* \times III^*$ collisions. The gauge groups and matter content that appear for such collisions are given in the following table.

Table 6: Gauge groups and matter content for $J = 1$ collisions.

Collision	Resolution	Gauge groups
$I_0^* \times III^*$	I_0^*, III, I_0, III^*	$SO(8), SO(7)$ or $G_2 \times SU(2) \times E_7$
$III^* \times III^*$	$III^*, I_0, III, I_0^*, III, I_0, III^*$	$E_7 \times SU(2) \times SO(7) \times SU(2) \times E_7$

Here we give the minimal allowed gauge groups.

- $I_0^* \times III^*$ collision ($I_0^* \times E_7$). This is the first case where it is not enough to make merely one blowup. One first has to blow up the collision which leads to type III singularity on the exceptional divisor. Singularities III and III^* are dual to each other and one has to make another blowup introducing one extra component of the exceptional divisor. Singularity I_0^* corresponds to either $SO(8)$, $SO(7)$ or G_2 , depending on whether the singularity is split, supersplit, or nonsplit. For simplicity we choose the “colliding” group to be G_2 . The gauge group appears to be $G_2 \times SU(2) \times E_7$ with the matter content $(3n_1 + 6)(\mathbf{7}, \mathbf{1}, \mathbf{1}) + (\frac{1}{2}n_2 + 3)(\mathbf{1}, \mathbf{1}, \mathbf{56})$ and the mixed representations

$$\frac{1}{2}(\mathbf{7}, \mathbf{2}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{2}, \mathbf{1}) \quad (4.8)$$

In this case we get two extra tensor multiplets. It is instructive to check the dimension of the hypermultiplet moduli space (for simplicity we consider the base being $P^1 \times P^1$). The number of independent parameters in polynomials f, g is $7 \cdot 6 + 10 \cdot 8 - 4 = 118$. The total number of $n_H - n_V = 273 - 29n_T$. Taking into account that we have two extra tensor multiplets we get an identity

$$118 + 6 \cdot 7 + 7 + 1 + 3 \cdot 56 - 14 - 3 - 133 = 186 . \quad (4.9)$$

In this example the full symmetry group is given by $G_2 \times SU(2) \times E_7$. Each factor has its own coupling constant, governed by the area of the corresponding divisor (1.2). Consider

specific regime, when G_2 coupling approaches zero, keeping $SU(2)$ coupling finite. In this regime we recover the $SU(2)$ gauge theory with *global* G_2 symmetry similar to the examples of phase transitions discussed in [14].

- $III^* \times III^*$ collision ($E_7 \times E_7$). In this case one needs to make five blowups. The first one leads to a I_0^* singularity on the exceptional divisor. In turn, as it was discussed above, the collision $III^* \times I_0^*$ requires two blowups of the base which produce additional components of the exceptional divisor with type III and I_0 singularities. The gauge group $E_7 \times E_7$ gets enhanced by a factor $SU(2) \times SO(7) \times SU(2)$, which follows from the structure of the singular locus. The matter content is given by $\frac{1}{2}(n_i + 5)\mathbf{56}$ and

$$\frac{1}{2}(\mathbf{1}, \mathbf{2}, \mathbf{8}, \mathbf{1}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{1}, \mathbf{8}, \mathbf{2}, \mathbf{1}). \quad (4.10)$$

Let us check the dimension of the hypermultiplet moduli space. The number of independent parameters in the polynomials f and g is $6^2 + 8^2 - 4 = 96$. Taking into account that we have two extra tensor multiplets (i.e. $n_H - n_V = 99$), we get an identity

$$96 + 2 \cdot 8 + 2 \cdot \frac{5}{2} \cdot 56 - 2 \cdot 3 - 21 - 2 \cdot 133 = 99. \quad (4.11)$$

It is worth mentioning that in both these examples we identify type III singularity with $SU(2)$ gauge group.

Note that by giving expectation values to the scalar components of universal part of matter one can higgs the gauge group on the exceptional divisor down to $SU(3)$ without any matter. This $SU(3)$ group cannot be higgsed further.

4.4. $J = 0$ collisions

The collisions $II \times I_0^*$, $II \times IV^*$ and $IV \times I_0^*$ do not require the blowup of the base. All other collisions require the blowup of the base, in some cases several times. In the table below we summarize the results on those collisions that can be resolved, preserving the Calabi-Yau condition.

Table 7. Gauge groups and matter for $J = 0$ collisions.

Collision	Resolution	Gauge groups
$I_0^{*ns} \times IV^{*ns}$	I_0^*, II, IV^*	$G_2 \times F_4$
$IV^{*ns} \times IV^{*ns}$	IV^*, I_0, IV, I_0, IV^*	$F_4 \times SU(3) \times F_4$

- $I_0^* \times IV^*$ collision. In this case one need to make one blowup of the base. For simplicity we consider a collision of *generic* singularities, which corresponds to $G_2 \times F_4$ gauge group. From the anomaly factorization conditions we get the following matter content $(7 + 3n_1)(\mathbf{7}, \mathbf{1}) + (4 + n_2)(\mathbf{1}, \mathbf{26})$ without any mixed representations.

The number of parameters in the polynomials f and g is equal to 128. Therefore we have

$$128 + 49 - 14 + 4 \cdot 26 - 52 = 215 \quad (4.12)$$

that coincides with $273 - 2 \cdot 29$. Note that this collision is related by the usual Higgs mechanism to $IV \times IV^*$ ($A_2 \times E_6$), which leads to the $SU(3) \times F_4$ gauge group.

• $IV^* \times IV^*$ collision. In this case one needs to make three blowups of the base. For simplicity we consider a collision of *generic* singularities, which corresponds to the $F_4 \times F_4$ gauge group. This gauge group get enhanced by an extra $SU(3)$ factor. (The case of $E_6 \times E_6$ can be considered in a similar way.) It is interesting that type IV singularity on one of the components of the exceptional divisor corresponds to $SU(3)$ group *without matter* and thus it could not be higgsed down. More specifically, from the anomaly factorization conditions we get the following matter content $(3 + n_1)(\mathbf{1}, \mathbf{1}, \mathbf{26}) + (3 + n_2)(\mathbf{26}, \mathbf{1}, \mathbf{1})$. Again, the dimension of the hypermultiplet moduli space

$$113 + 6 \cdot 26 - 2 \cdot 52 - 8 = 157 \quad (4.13)$$

agrees with $n_H - n_V = 273 - 4 \cdot 29$.

5. Tensionless strings and phase transitions

We found that resolution of colliding singularities sometimes leads to enhanced gauge symmetry. It is interesting to summarize possible gauge groups that appear on the exceptional divisor. There is an infinite series of examples where the gauge group gets enhanced by $Sp((n+m)/2)$ factor with $(n+m+8)(\mathbf{n} + \mathbf{m})$ matter multiplets. We also found $SU(2)$ with four doublets, $SU(3)$ without any matter and $SO(7)$ with two spinors.

When the area of the blown up sphere tends to zero the corresponding gauge theory approaches the strong coupling regime. It appears that $Sp((n+m)/2)$ gauge theory with $(n+m+8)(\mathbf{n} + \mathbf{m})$ matter multiplets is in the same universality class as F-theory compactification on elliptic Calabi-Yau manifold with the base being \mathbf{F}_1 . This theory exhibits phase transition which is due to small E_8 instantons (see ref. [2]). In fact, all resolutions with one blowup ($\delta n_T = 1$) discussed in this paper are in the same universality class. They can be continuously deformed into each other by adjusting some hypermultiplets (higgsing and unhiggsing). In this process one can break the gauge groups completely, keeping the possibility of making blow (-up or -down) intact. In this case we end up with the point-like E_8 instanton without any gauge group on top of it.

The other phase transitions are new⁶. Indeed the phase transition with $\delta n_T = 2$ is very different from the above ones. It corresponds to $SU(2)$ gauge theory with four doublets ($I_0^* \times III^*$ collision, two blowups). The spectrum formally coincides with that of

⁶ We are grateful to Cumrun Vafa for stimulating discussion on phase transitions and small instantons.

F-theory compactification on elliptic Calabi-Yau manifold with the base being \mathbf{F}_2 . This is consistent with the fact that in case of \mathbf{F}_2 it is known that one cannot go to a new phase by making one blowup (blowdown). The careful analysis of the elliptic fibration implies at the very last stage of the higgsation process we end up with $SU(2)$ gauge group with four doublets. The dimension of the Higgs branch is $8 - 3 = 5$. At the same time we have only 4 parameters in the elliptic fibration to relax in order to break $SU(2)$, keeping the possibility of making two blowups on top of each other. One extra parameter appears when we relax the condition that two blowup are on top of each other. This observation is consistent with the fact that two Calabi-Yau spaces, one elliptically fibered over the base with two blowup at *different points* in the base and the other elliptically fibered over the base with two blowup *on top of each other* are related by complex deformations. Therefore this phase transition can be described by two small E_8 instantons either at separate points or on top of each other depending on whether we keep $SU(2)$ gauge group broken or unbroken.

The case of $SU(3)$ gauge theory ($IV^{ns*} \times IV^{ns*}$ collision) requires three blowups. The gauge group $SU(3)$ is generic and cannot be higgsed away. The remaining case of $SU(2) \times SO(7) \times SU(2)$ ($III^{ns*} \times III^{ns*}$ collision) requires five blowups. The gauge group can be higgsed down to $SU(3)$. We believe that last two cases represent new universality classes of phase transitions. The interpretation of these phase transitions in terms of small instantons deserves further studies.

It is also worth mentioning that in the case of several blowups we get tensionless strings of different types, each corresponding to 3-branes wrapped around vanishing 2-cycles. The blown up cycles have a nontrivial intersection matrix which implies that nearly tensionless strings interact with each other. That presumably means that tensor multiplets also have nontrivial coupling.

It is interesting to note that in the process of blowing down one shrinks a 4-cycle (fibration of the singular fiber over the blown down P^1 , or collection of P^1 s) to a singular fiber. It is tempting to suggest that this 4-cycle is related to generalized del Pezzo surface, but the precise relation is unclear⁷.

So far we have analysed the collisions that can be resolved by blowing up the base without any violation of the Calabi-Yau condition. As mentioned above, these collisions lead to the phase transitions to new branches with different numbers of tensor multiplets, and hence they are closely related to the dynamics of tensionless strings. However, there are possible collisions of different type for which a resolution of the base would violate the Calabi-Yau condition. Here, we discuss some aspects of these collisions. For convenience we summarize possible collisions in the table below. How many blowups should be done is indicated by the number. The sign “—” means that the blowup procedure violates Calabi-Yau condition.

⁷ We are grateful to Sheldon Katz for the communications on this point.

Table 8. Possible collisions.

	I_0^*	$I_{n>0}^*$	I_n	II^*	II	IV^*	IV	III^*	III
$I_0^*, J = \text{any}$	1	1	0	4–	0	1	0	2	0
$I_{n>0}^*, J = \infty$	1	1	0						
$I_n, J = \infty$	0	0	0						
$II^*, J = 0$	4–			13–	1	6–	3–		
$II, J = 0$	0			1	3–	0	1–		
$IV^*, J = 0$	1			6–	0	3	1		
$IV, J = 0$	0			3–	1–	1	3–		
$III^*, J = 1$	2							5	1
$III, J = 1$	0							1	1

It is instructive to consider the scheme of resolutions of collisions which violate the Calabi-Yau condition

- $I_0^* \times II^* \rightarrow I_0^*, IV, I_0^*, II, I_0, II^*$
- $IV^* \times II^* \rightarrow IV^*, II, I_0^*, IV, I_0^*, II, I_0, II^*$
- $II^* \times II^* \rightarrow II^*, I_0, II, I_0^*, IV, I_0^* II, IV^*, II, I_0^*, IV, I_0^*, II, I_0, II^*$

For simplicity we do not consider collisions which involve exceptional singularities II, III, IV . It is curious to note that in all above three cases, the Calabi-Yau condition is violated at the step where one has to resolve the $II \times IV$ collision, which cannot be avoided upon the resolution. The physical meaning of the singularities II and IV in this situation is unclear, even so we have assigned $SU(3)$ gauge group to IV and nothing to II when they appear in the collisions discussed in the previous section. The $II \times IV$ collision corresponds to highly singular Calabi-Yau and does not lead to any Coulomb branch similar to those discovered in this paper. This suggests that the nonperturbative dynamics at this collision is not exhausted by tensionless strings and may imply an appearance of new physics.

On the other hand it seems that, say, in $II^* \times II^*$ collision, it is not necessary to go through the whole chain of prescribed resolutions. Let us suppose, for example, that we make just one first resolution. In this case we get the collision $II^* \times IV^* \times II^*$. In general this collision requires further resolutions of the base. Instead we may try to smooth out IV^* singularity by complex deformations. Thus we described a phase transition to a new branch with one new tensor multiplet. Again, this transition is governed by tensionless strings. However, in this case we are not able to describe the physics of the phase transition in terms of nonanomalous local field theory.

A similar procedure of smoothing and singularizing elliptic fibration by complex deformations can be used to make a number of consequent phase transitions.

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